Exercise 62

Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

Solution

Take the derivative of the given function,

$$\begin{aligned} y' &= \frac{d}{dx}(x^2 - 1) \\ &= \frac{d}{dx}(x^2) - \frac{d}{dx}(1) \\ &= (2x) - (0) \\ &= 2x, \end{aligned}$$

and evaluate it at $x = -1$.

$$y'(-1) = -2$$

This is what the slope of the parabola is at $x = -1$. The slope of the normal line is the negative reciprocal, $1/2$, and if it goes through the point $(-1, 0)$, it has the equation

$$\begin{aligned} y - 0 &= \frac{1}{2}(x + 1) \\ y &= \frac{1}{2}(x + 1). \end{aligned}$$

To find the values of x where this line intersects the parabola, set the two functions equal to each other and solve for x.

$$\begin{aligned} \frac{1}{2}(x + 1) &= x^2 - 1 \\ x^2 - \frac{1}{2}x - \frac{3}{2} &= 0 \\ (x + 1)\left(x - \frac{3}{2}\right) &= 0 \\ x &= \left\{-1, \frac{3}{2}\right\} \end{aligned}$$

Plug $x = 3/2$ into the given function for the parabola to find the corresponding y-value on the curve.

$$y\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 1 = \frac{5}{4}$$

Therefore, the normal line intersects the parabola a second time at

$$\left(\frac{3}{2}, \frac{5}{4}\right).$$

Below is a plot of the parabola and its normal line at $(-1, 0)$.